

SPY 1 to 16 October 2024

trades on 16 October 2024	312,181
average trading rate	17.1 per second
presample for the diurnal profile	17 to 30 September 2024
outliers on 16 October 2024, o	2,872 (0.92%)
price half-life: run / minimizer ¹	3.0 / 1.1 trades
cost of running 3 trades	+12.94%
scale half-life: run / minimizer ²	240 / 360 s
cost of running 240 s	+0.01%
direction half-life ³	37 s [28, 53]
Mandelbrot a minimizing QLIKE	72 trades
a within 0.0005 of the minimum	49 to 107
linear $\hat{\sigma}$, before $\rightarrow$ at the largest residual ⁴	6% $\rightarrow$ 875%
Huber $\hat{\sigma}$, before $\rightarrow$ peak	6% $\rightarrow$ 13%
median spot volatility, Huber / linear input ⁵	7.5 / 9.9 %
hyperbolic against HExp, correlation of logs	0.9927
HExp QLIKE advantage, leave-one-day-out ⁶	+0.00213
15-minute signature level	6 to 17 %
Huber signature, 1 min over 1 s, 12 day average	1.18

SPY over the 12 trading days 1 to 16 October 2024, and on 16 October 2024. ¹the median over k from 2 to 100 of the half-life minimizing the k -trade-ahead Huber forecast loss, and the largest cost over those k of running the common 3 trades. ²the 1-minute-ahead check loss of the noise scale. ³the half-life maximizing the mean predictive log-likelihood per scored trade on 16 October 2024, with the range within 10^{-4} of the maximum. ⁴the largest Huber residual after 09:45, at 10:23; before is the median over $t^* - 15$ to $t^* - 3$ minutes and at is the maximum over $t^* - 6$ to $t^* + 12$ minutes. ⁵over the 12 days, sampled every 300 trades. ⁶hyperbolic minus HExp, so positive favours HExp.

k	Mean of $ \hat{X}_t - \hat{X}_{t-k} /\sqrt{k}$ in cents				Sqrt of Mean of $(\hat{X}_t - \hat{X}_{t-k})^2/k$ in cents			
	median	Huber	linear	trades	median	Huber	linear	trades
1	0.160	0.123	0.142	0.529	0.508	0.312	2.374	14.778
2	0.170	0.148	0.174	0.457	0.462	0.320	2.254	10.457
5	0.211	0.188	0.219	0.379	0.414	0.319	1.952	6.622
10	0.241	0.221	0.248	0.344	0.394	0.335	1.600	4.690
25	0.266	0.254	0.270	0.316	0.378	0.352	1.100	2.979
k	ACF of $\hat{X}_t - \hat{X}_{t-1}$ at lag k				Hill tail index of $ \hat{X}_t - \hat{X}_{t-k} $			
	median	Huber	linear	trades	median	Huber	linear	trades
1	-0.17	0.05	-0.10	-0.50	2.01	1.71	1.62	1.76
2	-0.02	-0.05	-0.08	0.00	1.89	1.59	1.61	1.81
5	0.01	0.04	-0.04	0.00	2.05	1.85	1.57	1.83
10	0.00	0.00	-0.01	0.00	2.50	2.45	1.68	1.87
25	0.00	0.00	0.00	0.00	3.59	3.94	2.09	1.97

Table 1: Trade-lag differences, SPY, 16 October 2024.

s	Mean of $\frac{ \hat{X}_t - \hat{X}_{t,s} }{\sqrt{\tau_t - \tau_{t,s}}}$ in cents				Annualized volatility: $\hat{\sigma}_{\text{Annual};s}$			
	median	Huber	linear	trades	median	Huber	linear	trades
0.1 s	1.77	1.58	1.71	2.73	4.7	3.5	44.3	168.8
1 s	1.37	1.30	1.38	1.72	5.0	4.6	24.9	107.0
5 s	1.31	1.29	1.32	1.41	5.5	5.4	9.9	26.9
10 s	1.28	1.27	1.32	1.50	5.6	5.5	8.6	30.1
1 min	1.27	1.27	1.27	1.30	5.7	5.7	6.1	10.7
15 min	1.30	1.30	1.30	1.30	6.1	6.1	6.2	6.4
s	Cor($\hat{X}_t - \hat{X}_{t,s}, \hat{X}_{t,s} - \hat{X}_{t,2s}$)				Hill tail index of $ \hat{X}_t - \hat{X}_{t,s} $			
	median	Huber	linear	trades	median	Huber	linear	trades
0.1 s	-0.09	-0.05	-0.27	-0.08	4.11	4.00	3.08	2.02
1 s	0.01	0.06	-0.46	-0.66	6.81	7.04	2.73	1.93
5 s	0.02	0.04	-0.18	-0.01	9.23	9.70	4.16	2.58
10 s	0.05	0.06	-0.77	-0.95	8.23	8.10	3.43	2.23
1 min	0.05	0.05	-0.07	-0.71				

Table 2: Clock-lag differences, SPY, 16 October 2024.

	median	Huber	linear
Mean of $ \hat{\epsilon}_t $ in cents	0.40	0.51	0.55
s.d. of $\hat{\epsilon}_t$ in cents	10.46	10.45	8.73
Tail index of $\psi(\hat{\epsilon}_t)$	∞	∞	1.62
ACF of $\psi(\hat{\epsilon}_t)$, lag 1	0.31	0.42	-0.10
ACF of $\psi(\hat{\epsilon}_t)$, lag 2	0.14	0.29	-0.08
ACF of $\psi(\hat{\epsilon}_t)$, lag 5	0.06	0.14	-0.04
ACF of $\psi(\hat{\epsilon}_t)$, lag 10	0.03	0.04	-0.01
ACF of $\psi(\hat{\epsilon}_t)$, lag 25	0.01	0.01	0.00
80% C.I. for $\theta_t - X_t$	—	-0.42,0.42	-0.43,0.44
95% C.I. for $\theta_t - X_t$	—	-0.75,0.72	-0.81,0.79
99% C.I. for $\theta_t - X_t$	—	-1.13,1.07	-1.36,1.39
s.e. of $\theta_t - X_t$ in cents	—	0.35	$\infty(2.24)$

Table 3: Residual summaries, SPY, 16 October 2024. Median C.I. and s.e. cells are omitted (price discreteness); the linear s.e. does not exist in theory, the sample value is in brackets.

	16 October 2024		12 days, min to max	
	upper	lower	upper	lower
All trades (as in the paper)	1.62	1.39	1.13 to 1.69	1.26 to 1.63
Residual divided by δ_t	1.51	1.39	1.13 to 1.64	1.27 to 1.63
Repricing windows removed	1.61	1.39	1.11 to 1.68	1.24 to 1.69
Calm windows only	1.59	1.33		
Every 10th trade	1.64	1.55	1.16 to 1.67	1.31 to 1.86
Six-trade half-life	1.77	1.53	1.22 to 1.92	1.36 to 1.88

Table 4: Hill estimator of the tail index of the Huber filter residual, SPY, 12 analysis days. The index is the mean of Hill estimates over the largest 0.1% to 1% of residuals, each tail separately, which for this stock is an average over 13 to 14 depths a day in row 1; the masked and thinned rows use fewer. Second row: divides the residual by its own threshold δ_t . Third row: drops trades where the filtered price moves more than 6.8 times the day’s median δ_t within ± 1 second, which is 17.9 to 20.2 cents over the days and removes 0% to 5% of a day; the fourth keeps only trades where it moves less than 3.4 times that median within ± 5 seconds, 9.0 cents on 16 October 2024, 75% of that day. Fifth row: keeps every 10th trade. Sixth row: doubles the filter’s half-life with the thresholds held fixed. The ranges run over the days on which the row can be estimated.

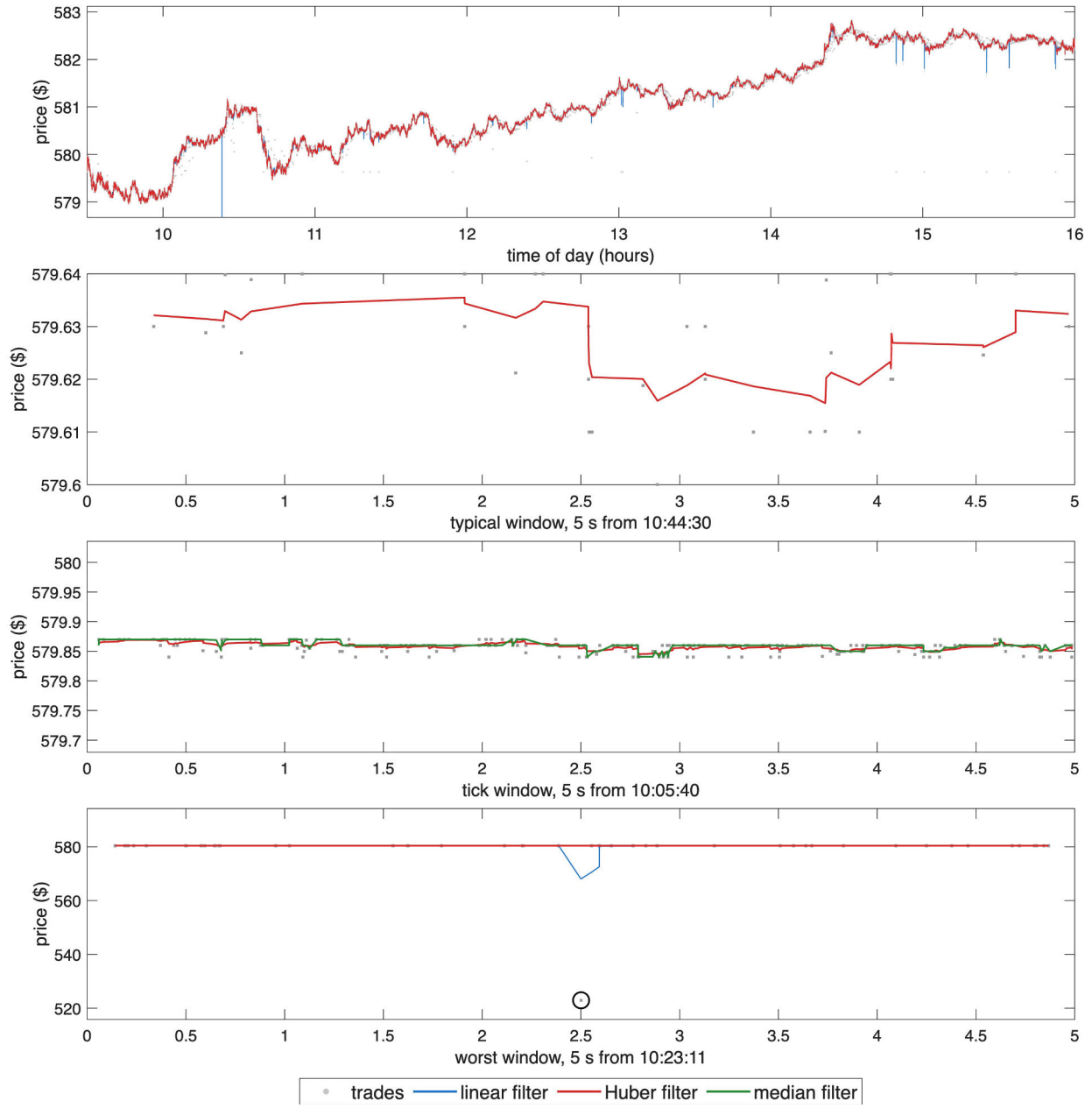


Figure 1: SPY trades (grey points), the linear filter and the Huber filter, 16 October 2024. Top: full trading day. 2nd: 5-second window at the day's median trade count, from 10:44:30. 3rd: 5-second window at 10:05:40 in which the price reverses direction most often, with the median filter m_t added. Bottom: 5-second window around the largest Huber residual after 09:45, circled.

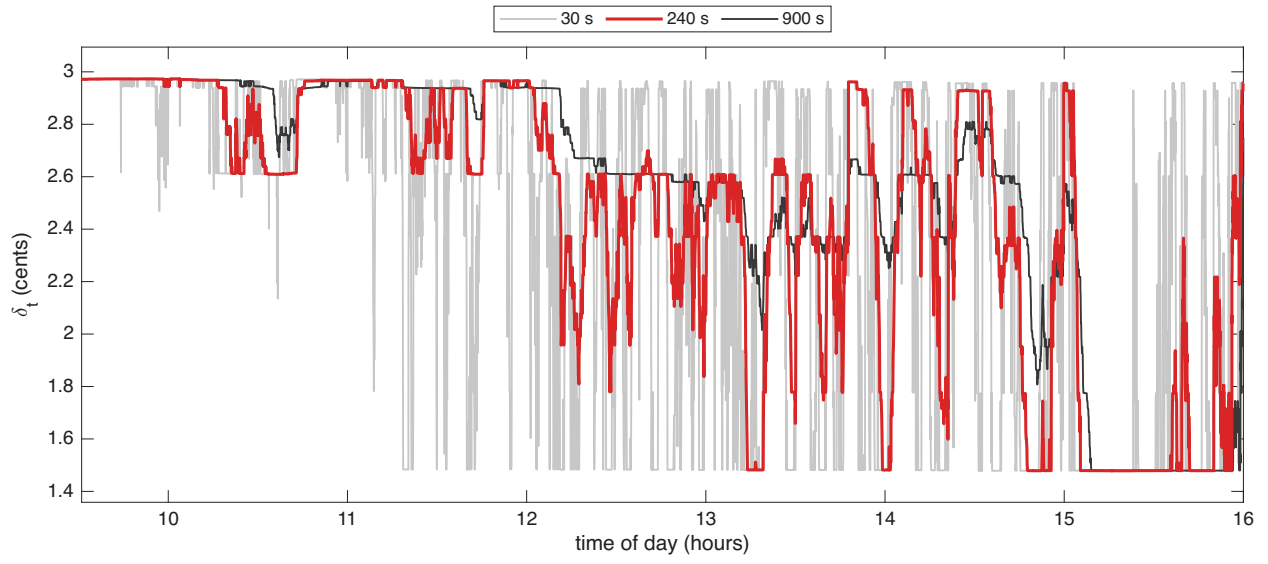


Figure 2: The Huber threshold δ_t in cents for SPY on 16 October 2024, with half-lives of 30, 240, 900 seconds. The 240-second half-life, in red, is used.

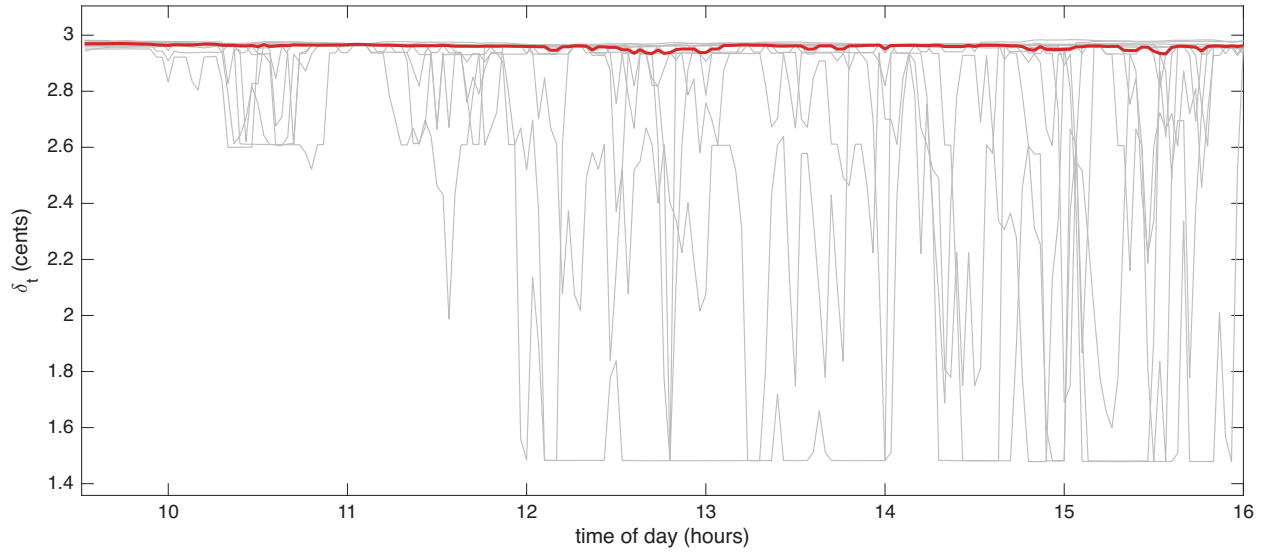


Figure 3: Huber threshold δ_t in cents through the trading day, for 12 SPY trading days in Oct 2024 (grey lines) and their median (red line).

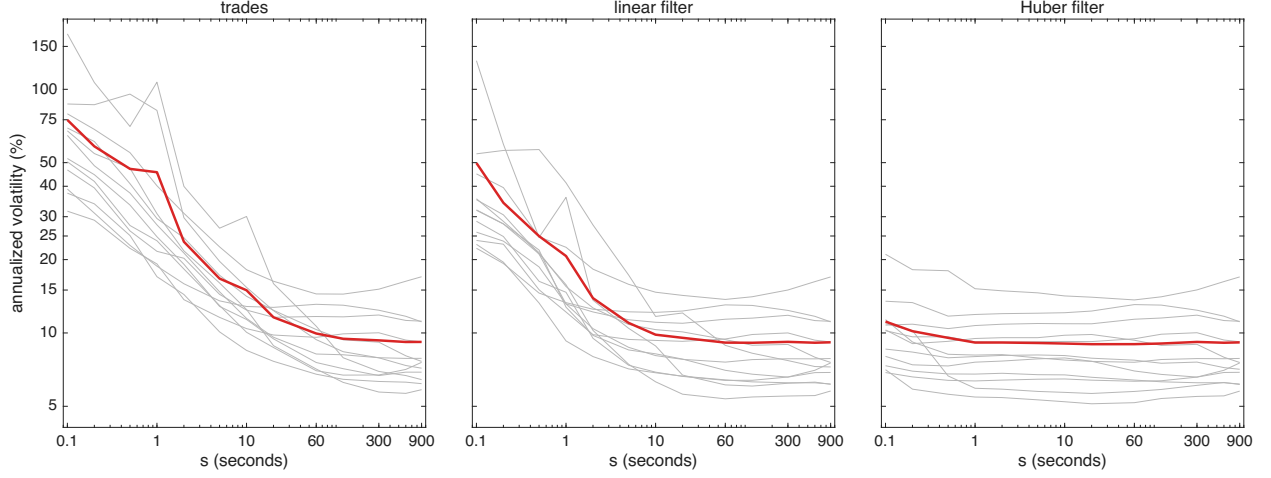


Figure 4: Volatility signature curves for SPY over the 12 trading days in Oct 2024. The annualized vol $\hat{\sigma}100\sqrt{252}$ from the returns $\hat{X}_t - \hat{X}_{t,s}$ of Table 2, against the lag s in seconds, both axes on log scales, for: raw trades, linear filter & Huber filter. Grey lines are the individual trading days & the red line is the square root of the 12 day average of their squares. The linear & Huber returns carry the finite-sample correction $c_{t,s}$ of the paper; the raw trades are unaffected by it.

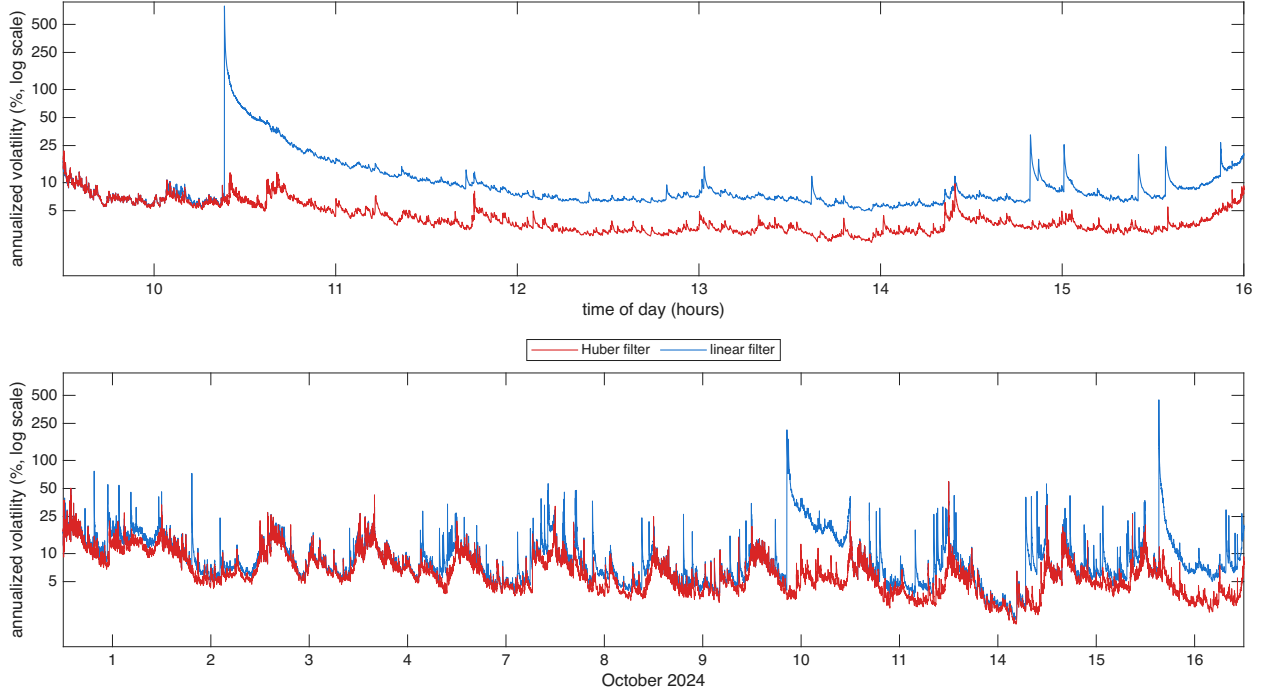


Figure 5: Filtered spot volatility for SPY over 12 trading days in Oct 2024, the diurnally adjusted estimator $\hat{\sigma}(u)100\sqrt{252}$ at $s = 1$ second with hyperbolic weights, Mandelbrot parameter $a = 72$, with the input prices computed using the Huber & linear filters, shown as annualized volatility in percent on a log10 scale. Top: 16 October 2024, bottom: 12 days.

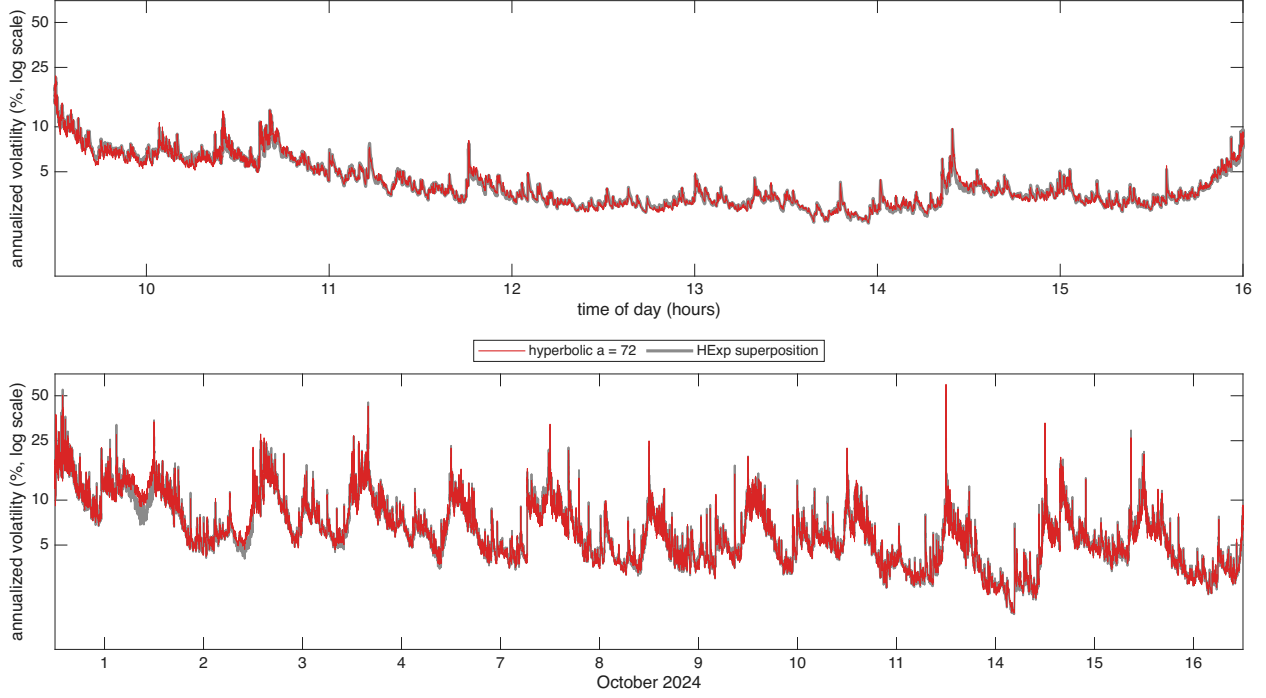


Figure 6: Hyperbolic against superposition weights for SPY over 12 trading days in Oct 2024, the diurnally adjusted $\hat{\sigma}(u)100\sqrt{252}$ at $s = 1$ second from the Huber filter, shown as annualized vol in % on a log10 scale. The only difference between the 2 paths is the weight function: the hyperbolic weights with $a = 72$ in red, drawn on top, and the HExp superposition in grey, 4 exponential components with half lives of 10, 100, 1000, 10000 seconds. Hyperparameters minimize the QLIKE objective. Top: 16 October 2024, bottom: 12 days.

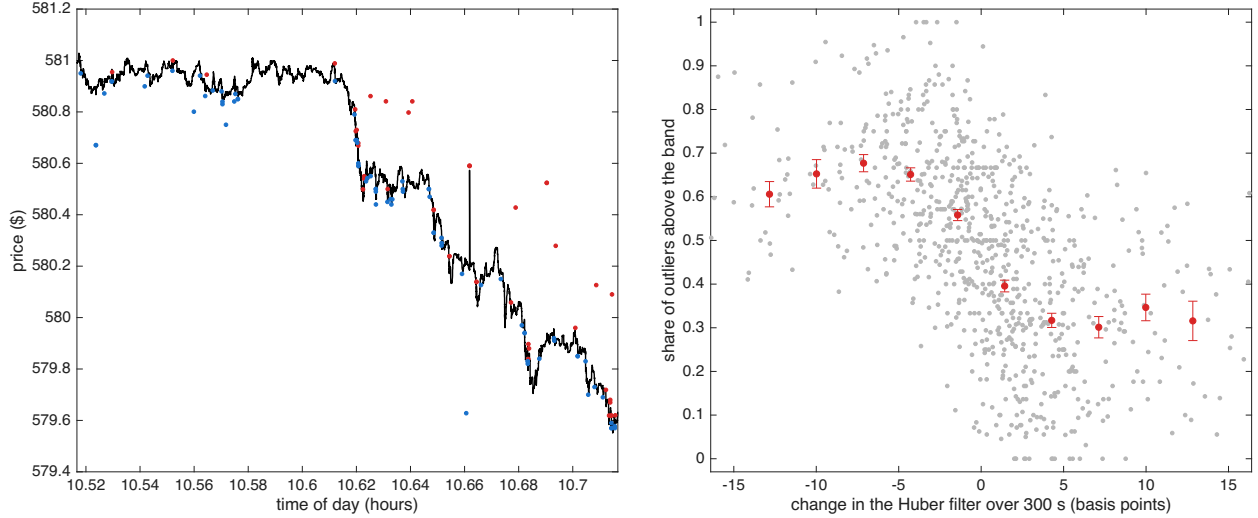


Figure 7: Outliers & robust price for SPY on 16 October 2024. Left: the 12 minutes with the largest move in the Huber filter that day; outliers above $Y_t - \theta_t > \delta_{t-1}$ in red, outliers below $Y_t - \theta_t < -\delta_{t-1}$ in blue, Huber filter in black. Right: over the 12 trading days, the 300 second empirical conditional probability that outliers are above (rather than below) the Huber band against the 300 second change in the Huber filter in basis points: light points are single bins, the line & bars are means & s.e.

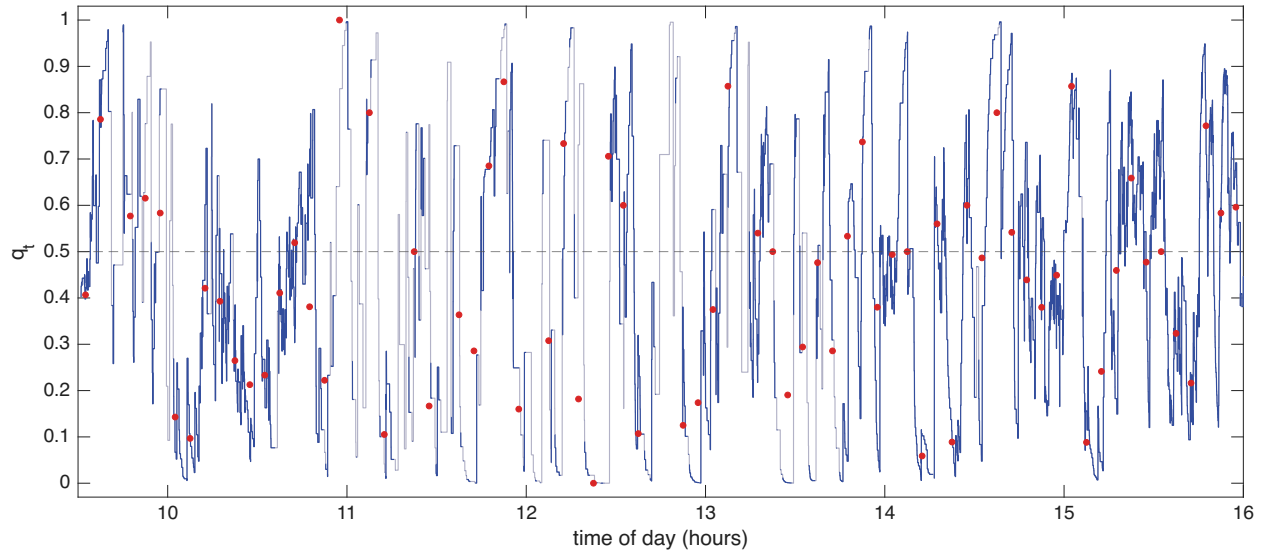


Figure 8: The directional conditional probability for *SPY*, 16 October 2024. The filtered conditional probability q_t that an outlier lies above the robust Huber band, under exponential weights decaying in clock time with half life of 37 seconds; where fewer than two outliers remain in the weights it is drawn in grey. The dashed line marks 0.5. The red points are the empirical conditional probability using the 300 second bins of the previous figure.

The common half-life of 3 trades raises the k-trade-ahead Huber forecast loss by 12.9% over the minimizing half-life of 1.1 trades.

At the horizons $k = 2, 3, 5, 10, 30$ and 100 the half-life minimizing the k-trade-ahead Huber forecast loss is 1.1, 1.1, 1.1, 1.1, 1.1 and 1.1 trades, within one grid point of the end of the 1 to 5 trade grid searched.

Selecting a on single days gives values from 13 to 562 trades, against 72 trades pooled over the 12 days; the 10-second estimate of the spot volatility behind the QLIKE objective averages roughly 171 trades for this stock.

1 return(s) share a timestamp with the opening auction and are left out of the volatility sums.

Diagnostics.